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OPTIMUM CONTOUR HEAT REJECTION FINS COOLED BY RADIATION

1962

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OPTIMUM CONTOUR HEAT REJECTION FINS COOLED BY RADIATION

Part II. In Ref.I(Points 1-2) a two - dimensional problem of a radiating fin of optimum contour was discussed (heat is assumed to be supplied from one side). Initial heat flow Q_o, temperature T_o, minimum fin thickness y_{min}.

In was shown in the paper that cross section area F of an optimum fin (of minimum weight) can be defined by

$$F = \frac{\bar{F}}{F_{opt}} \frac{Q_o^3}{\lambda \sigma^2 E^2 T_o^2} \qquad (4.0)$$

where λ - coefficient of conductivity

σ - Stefan - Boltzmann constant

E - the emissivity of the fin surface, $\frac{F}{F_{opt}}$

is defined by the ratio of y_{min} to initial fin thickness y_o

$$\frac{F}{F_{opt}} = 1.0 \text{ with } \frac{y_{min}}{y_o} = 0$$

Let us consider an optimum combination of radiating fins and a tube with inner heating, for wall temperature T_o and tube geometry prescribed.

(D - diameter of the tube, δ - wall thickness).

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Consider a combination of a tube and two fins, arranged on both sides of the tube in the same plane.

Mutual irradiation of the tube and fins is neglected in the first approximation. Then total amount of heat, rejected per unit length is

$$\frac{dq}{dZ} = 2Q_o + \pi(D+\delta)\,\sigma\,\varepsilon\,T_o^4 \qquad (4.1)$$

Unit weight of the tube with fins is equal to

$$\frac{dG}{dZ} = \frac{\pi(D+\delta)^2 - \pi D^2}{4}\gamma_1 + \frac{2\frac{F}{F_{opt}}}{\lambda\sigma^2\varepsilon^2}\cdot\frac{Q_o^3}{T_o^9}\gamma_2 \qquad (4.2)$$

where γ_1 - tube material specific weight

γ_2 - fin material specific weight

From 4.1 and 4.2 the specific weight of the tube with fins $\frac{dG}{dq}$ is

$$\frac{dG}{dq} = \frac{A + BQ_o^3}{Q_o + C} \qquad (4.3)$$

where $A = \pi\frac{(D+\delta)^2 - D^2}{8}\gamma_1$; $B = \frac{\frac{F}{F_{opt}}}{\lambda\sigma^2\varepsilon^2 T_o^9}\gamma_2$

$$C = \frac{\pi(D+\delta)}{2}\sigma\varepsilon T_o^4$$

Minimum specific weight of the combination is obtained

when

$$Q_o^2 = \frac{A}{B(2Q_o + 3C)}$$

or

$$(1-\chi)\chi^2 = \frac{4}{27}\,\frac{A}{BC^3}$$

where

$$\chi = \frac{2}{3}\frac{Q_o}{C} \qquad (4.4)$$

then minimum specific weight is

$$\left(\frac{dG}{dQ}\right)_{min} = \frac{A/C}{1+\chi} \qquad (4.5)$$

i.e. in $\frac{1}{1+\chi}$ times less than that of a tube without fins.

Values of $\frac{1}{1+\chi}$ are presented (Fig.1) for $D = 10$ mm,

$\delta = 0.25 \div 0.75$ $T_o = 1100^\circ K$, $E = 0.9$ (the tube made of steel, fins - of steel, copper and berillium for $\frac{F}{F_{opt}} = 1.205$,

an optimum contour with $\frac{y_{min}}{y_o} = 0.31$ (see 2).

If the optimum fins discussed are replaced by those of optimum rectangular cross section for $\frac{F}{F_{opt}} = 1.635$ (see above)

their specific weight is increased by $\sim 5\%$ (Fig.1).

5. From formula /4.0/ cross section area F of an optimum fin and its weight are proportional to the cube of heat flow Q_o radiated by the fin.

Using "n" radiating fins to remove heat flow Q_ε (mutual irradiation is neglected) total cross-section area (and their weight too) will decrease inversely proportionate with respect to n^2

$$F_\varepsilon = nF \sim \frac{1}{n^2} \quad \text{with} \quad Q_\varepsilon = nQ_o = \text{const} \qquad (5.0)$$

Radiating fins can be arranged so as to form a starwise contour at the apex of a polyhedron, cooled by radiation.(Fig.2).

Here for $n > 2$ mutual irradiation of fins is of importance. It being taken into account, an optimum number of "n" fins and the corresponding optimum fin cross section contour are determined.

Consider a two-dimensional problem of defining an optimized fin contour used to remove heat from a polyhedral prism (mutual irradiation of its edges and surfaces is taken into consideration).(2).

We analyze thin fins of shallow contour, for which thermal radiation law and the equation of heat transfer are true in the form of

$$\frac{1}{2} Q(X) = -\lambda y(X) \frac{dT}{dX}; \quad -\frac{1}{2} dQ = q_{pes}(X)\, dS \qquad (5.1), (5.2)$$

where dS - an element of length of the fin side surface

$q_{pes}(X)dS$ - the difference between heat flow absorbed by all surfaces "seen" from dS, and that absorbed by dS. (the surrounding medium has parameters $\varepsilon = 1$, $T = 0$).

The difference characterizes the amount of heat, radiated by the element dS (mutual irradiation of adjacent fins is taken into consideration).

As thin fins are discussed in this case, for determination of q_{pes} surfaces dS can be considered lying along the fin axis (Fig.3).

For simplying the analysis let us take the case $\varepsilon = 1$ for all surfaces. In the problem $q_{pes}(x)$ is expressed as follows (3) (with $\varepsilon = 1$ the reflected heat flow is absent):

$$q_{pes}(x) = \sigma \left(\frac{1-\sin\varphi_1(x)}{2} T^4(x) - \int_{\varphi_1(x)}^{\varphi_2(x)} T^4(\varphi) \frac{\cos\varphi}{2} d\varphi - \int_{\varphi_0(x)}^{\pi/2} T_0^4 \frac{\cos\varphi}{2} d\varphi \right) + \sigma \frac{1+\sin\varphi_1(x)}{2} T^4(x) \qquad (5.3)$$

Angles $\varphi(X)$, $\varphi_0(X)$, $\varphi_1(X)$ are illustrated in Fig.3.

Having expressed φ, φ_0, φ_1 through X and ξ rewrite eq. /5.3/

$$-\frac{1}{2}\frac{dQ}{dX} = -\frac{\sigma}{2}\left[2T^4(X) - F_1(X) \cdot T_0^4 - \int_{x_0}^{x_1} T^4(\xi) \frac{X\xi \sin^2\gamma \, d\xi}{(x^2+\xi^2-2X\xi\cos\gamma)^{3/2}} \right],$$

where

$$F_1(X) = 1 - \frac{X - X_0 \cos\gamma}{(X^2+X_0^2-2XX_0\cos\gamma)^{1/2}}$$

$\gamma = \frac{2\pi}{n}$ - angle between adjacent fins formulate a variational problem: to find a fin cross section of minimum area F (i.e. of minimum weight) for a polyhedral prism geometry, its surface temperature and net radiant heat flux W prescribed.

This is a problem of minimizing the functional

$$F = 2n \int_{x_o}^{x_1} y(X)\, dx \qquad (5.5)$$

under conditions

$$\frac{1}{2} Q(X) = \lambda y(X) \frac{dT}{dx} \qquad (5.1)$$

$$-\frac{dQ}{dX} = \sigma \left[2\, T^4(X) - F_{\xi}(X) T_o^4 - \int_{x_o}^{x_1} T^4(\xi) \frac{X \xi \sin^2 \gamma \, d\xi}{(X^2 + \xi^2 - 2X\xi \cos\gamma)^{3/2}} \right] \qquad (5.2)$$

$$W = n(Q_o + q_o) \qquad (5.6)$$

where q_o - resulting radiation from a side of the prism (its irradiation by fins being taken into account).

$$q_o = \sigma \int_0^{x_o \sin\gamma/2} \left[\int_{\psi_1(z)}^{\pi/2} \left\{ T_o^4 - T^4(\psi) \cos\psi \right\} d\psi + T_o^4 \left\{ \sin\psi_1(z) - \sin\psi_2(z) \right\} + \int_{\psi_2(z)}^{-\pi/2} \left\{ T_o^4 - T^4(\psi) \cos\psi \right\} d\psi \right] dz \qquad (5.7)$$

ψ and z illustrated (Fig.4).

Integrating 5.2 and using 5.1 and 5.6, it becomes possible to express the functional 5.5 as follows:

$$F = 2n \int_{x_o}^{x_1} \frac{B(X)}{T^1(X)} dx \qquad (5.8)$$

where

$$B(X)=\frac{1}{2\lambda}\left\{ \sigma\left[\int_{x_0}^{x} \left(2T^4(X)-F_1(X)T_0^4\right)dX\right. \right. -$$

$$\left. - \sin^2\gamma \int_{x_0}^{x_1} T^4(\xi)\left(\int_{x_0}^{x}\frac{xdx}{(X^2+\xi^2-2X\xi\cos\gamma)^{3/2}}\right)\xi d\xi - \left(\frac{W}{n}-q_0\right)\right\}$$

It is expedient to introduce dimensionless variables

$$\bar{Q}=\frac{Q}{W}\ ;\ \bar{T}=\frac{T}{T_0}\ ;\ \bar{q}_0=\frac{q_0}{W}\ ;\ \bar{X}=\frac{X}{X^*};\ \bar{y}=\frac{y}{y^*}\ ;\ \xi=\frac{\xi}{X^*};\ \bar{F}=\frac{F}{F^*}$$

where

$$W=\frac{W}{n}\ ;\ X^*=\frac{W}{2\sigma T_0^4}\ ;\ y^*=\frac{X^*W}{2\lambda T_0}\ ;\ F^*=\frac{W^3}{8\lambda\sigma^2 T_0^9}$$

In terms of new variable the functional takes the form

$$\bar{F}=\frac{2}{n^2}\int_{\bar{x}_0}^{\bar{x}_1}\bar{y}d\bar{x}=\frac{2}{n^2}\int_{\bar{x}_0}^{\bar{x}_1}\frac{\bar{B}(\bar{X})\,d\bar{x}}{\bar{T}^1(\bar{X})} \qquad (5.9)$$

and

$$\bar{B}(X)=1/2\int_{\bar{x}_0}^{\bar{x}}\left[2\bar{T}^4(\bar{X})-F_1(\bar{X})\right]d\bar{x}-\frac{1}{2}\int_{\bar{x}_0}^{\bar{x}_1}\bar{T}^4(\bar{\xi})\,G(\bar{X},\bar{\xi})d\xi-(1-\bar{q}_0)$$

$$G(\bar{X},\bar{\xi})=\frac{\bar{\xi}-\bar{X}_0\cos\gamma}{(\bar{X}_0^2+\bar{\xi}^2-2\bar{X}_0\bar{\xi}\cos\gamma)^{1/2}}-\frac{\bar{\xi}-\bar{X}\cos\gamma}{(\bar{X}^2+\bar{\xi}^2-2\bar{X}\bar{\xi}\cos\gamma)^{1/2}}$$

Thus the problem set here, is reduced to that of finding the function $\bar{T}(\bar{X})$, giving minimum to the functional 5.9.

Unknown function $\bar{T}(\bar{X})$ must satisfy the following boundary conditions: $\bar{T}(\bar{X}_0)=1$; $\bar{T}(\bar{X}_1)=\bar{T}_1$

if $\bar{T}$ fin tips temperature is prescribed and $\frac{\bar{B}(\bar{X}_1)}{\bar{T}'(\bar{X}_1)^2} = 0$ (5.10)

if the right-hand boundary condition is natural.

Besides, if fin width $\bar{X}_1 - \bar{X}_0$ is not prescribed, the optimum value of $\bar{X}_1$ must satisfy the condition:

$$\frac{\bar{B}(\bar{X}_1)}{\bar{T}^1(\bar{X}_1)} = 0 \qquad (5.11)$$

The condition is equivalent to that of 5.10, and it is generally characteristic of functionals having the form

$$J = \int_{a}^{b} \frac{F(y,x)}{y^1(x)}\,dx$$

This variational problem can be solved by the functional method of steepest descent

Note that

$$X_0 = \frac{2X_0 \quad \sigma T_0^4}{W} = \frac{1}{\sin \gamma/2} \quad \frac{W_0}{W}$$

where W_0 - heat flux, radiated by one side of a prism, having no fins.

Ratio $\frac{W_0}{W}$ - gives the portion of heat flux, radiated by a prism without fins with respect to that, radiated by a prism with fins.

When $\bar{X}_0$ is sufficiently small, the values of $\frac{W_0}{W}$

and $\bar{q}_0$ are also small.

When $\bar{X}_0 \to 0$; $\bar{q}_0 \ll 1$ (expression for $\bar{B}(\bar{X})$)the solution gives mutual irradiation of fins themselves.

Numerical examples for $T_1=0$ are presented (Fig.5).

The ratio of total cross section area of " n" fins to that of a single fin (n = 1) for the same heat parameter (W, T_0, λ) is presented in Fig.6.

$$\frac{\bar{F}}{\bar{F}_{1\ opt}} = \frac{1}{4\ n^2} \int_{\bar{x}_0}^{\bar{x}_1} \bar{y} d\bar{x}; \quad \bar{F}_{1\ opt} = 8$$

Solid line in Fig.6 shows the optimum solution (see Fig.5) the dash lines correspond to the results, obtained in work (6) for optimum rectangular fins of starwise arrangement.

Dash. - dotted line (Fig.6) corresponds to the correlation (5.0) (mutual irradiation of fins is not taken into account).

We see (Fig.6) that mutual irradiation of fins being taken into account, the least area is that of four optimum fins (n =4). The area of the system, consisting of [redacted] optimum rectangular fins is in ~1.5 times more than that of optimum fins.

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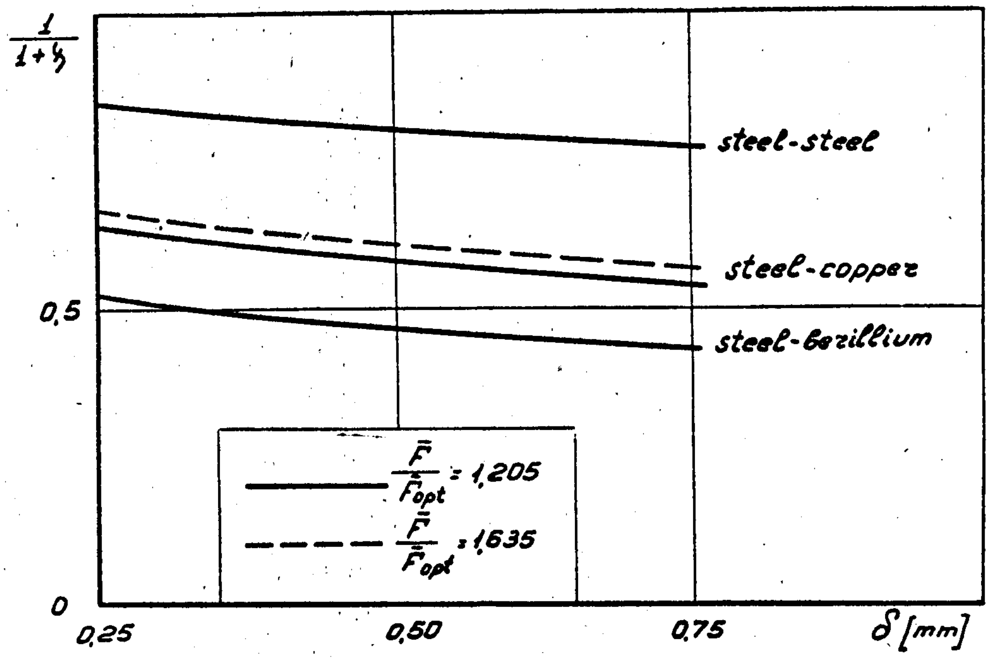

Fig. 1

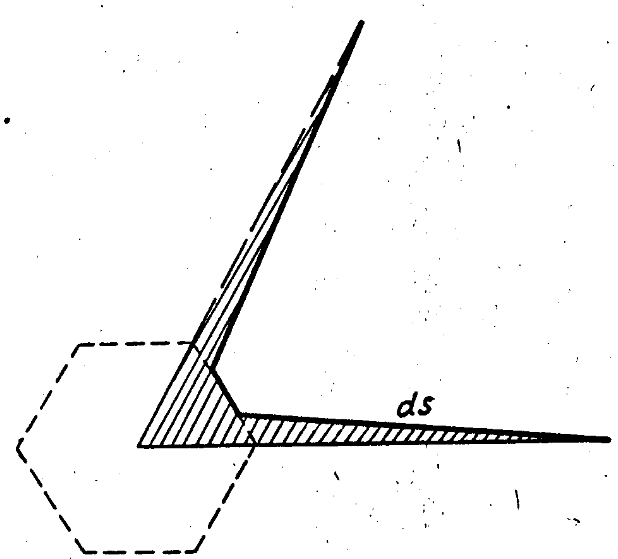

Fig. 2

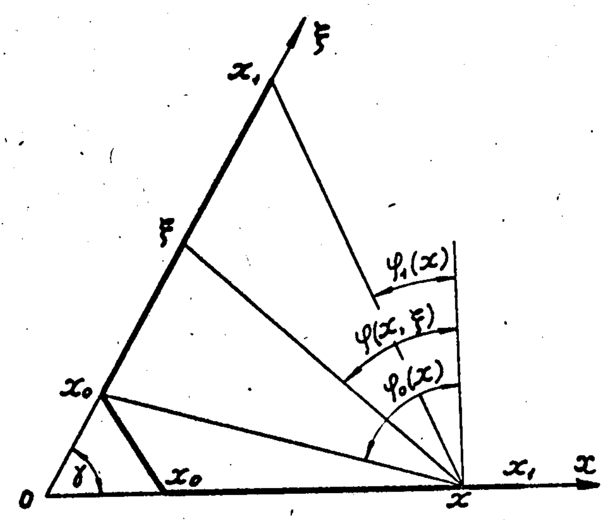

Fig. 3

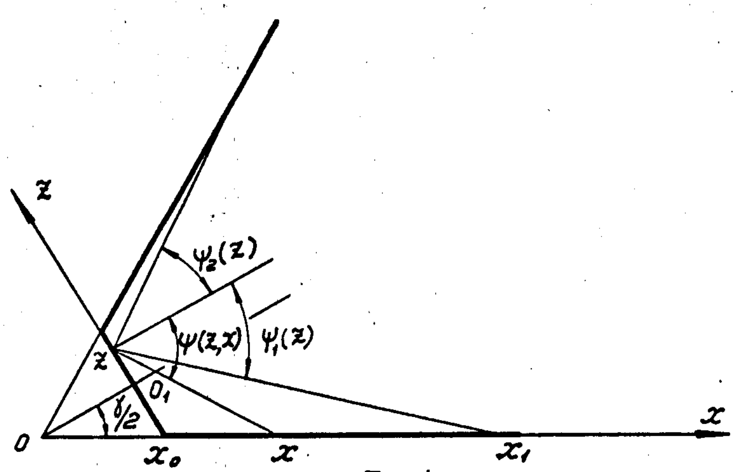
z
$\psi_2(z)$
$\psi(z,x)$
$\psi_1(z)$
z
0_1
$\gamma/2$
0
x_0
x
x_1
x

Fig. 4

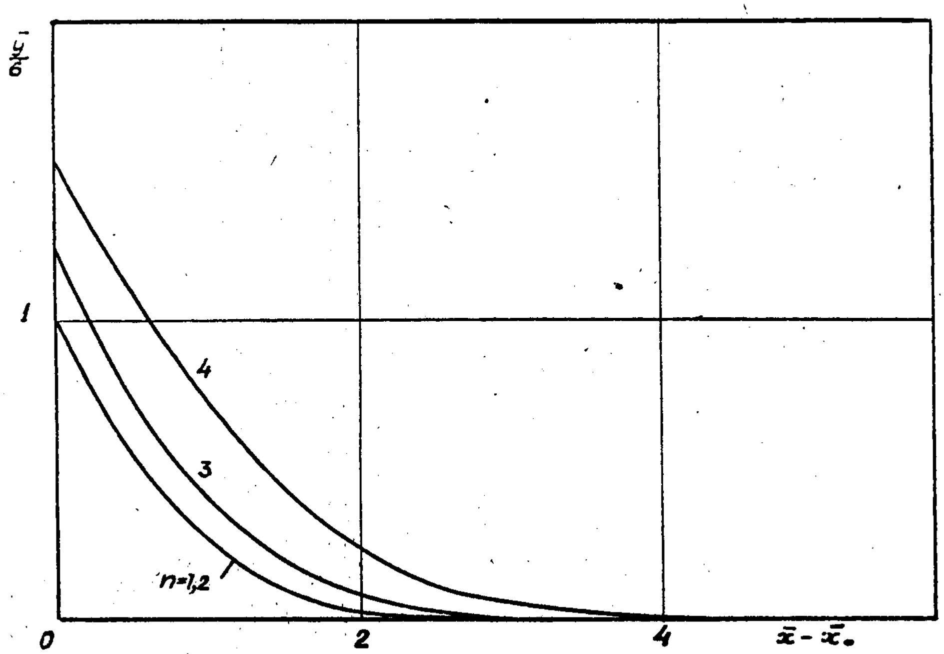
1
4
3
n=1,2
0
2
4
$\bar{x}-\bar{x}_0$

Fig. 5

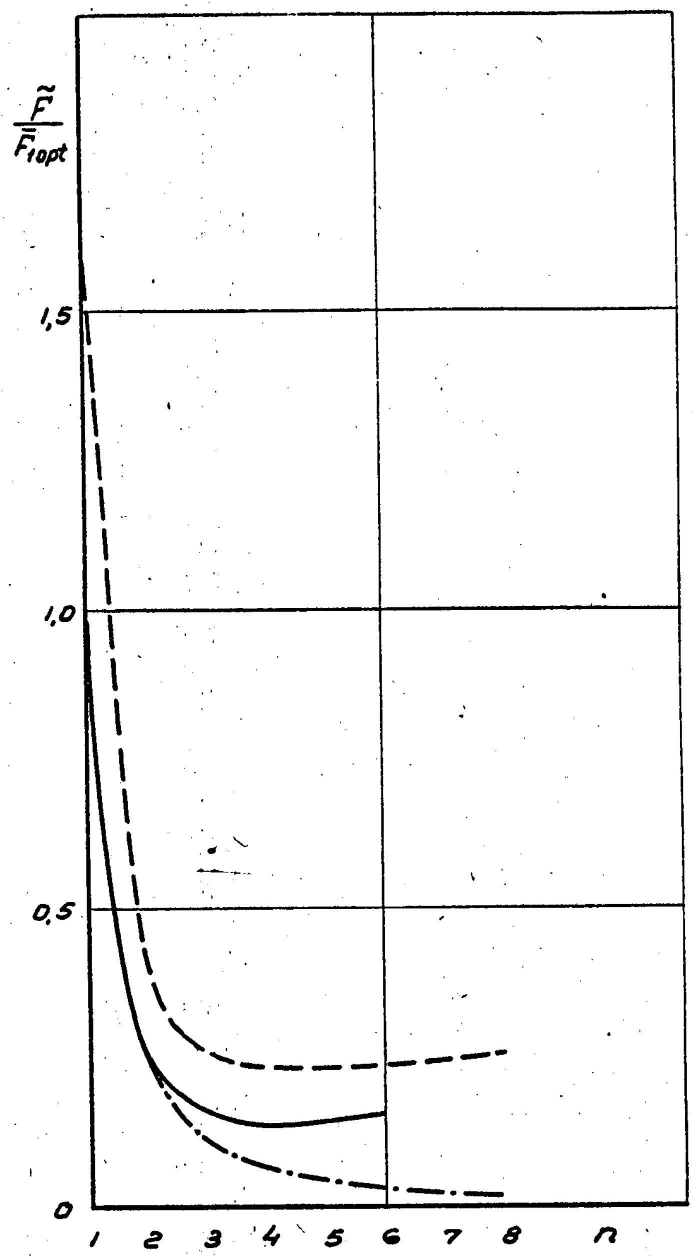

Fig. 6

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ON THE MOTION OF A BODY OF VARIABLE MASS WITH CONSTANT AND DECREASING POWER CONSUMPTION IN A GRAVITATIONAL FIELD

Part II

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ON THE MOTION OF A BODY OF VARIABLE MASS WITH CONSTANT AND DECREASING POWER CONSUMPTION IN A GRAVITATIONAL FIELD

Part II x)

by GRODZOVSKY G.L., IVANOV Y.N. and TOKAREV V.V. xx)

6. In sections 1-5 (see part I/1/) the general case of jet propulsion optimization of a body of variable mass in a gravitational field with constant power con-umption N=const and multistage power decrease with a corresponding proportional weight decrease of the power source $G_N = \alpha N$; α = const was considered.

By generalizing the problem given in section 1 we state the following variational problem /2/:

Let us assume, that the relative useful weight is prescribed $\bar{G}_n = \frac{G_n}{G_0} = 1 - \bar{G}_{N_0} - \bar{G}_{M0}$, where $\bar{G}_{N_0} = \frac{G_{N_0}}{G_0}$ and $\bar{G}_{M0} = \frac{G_{M0}}{G_0}$ (G_{N_0}, G_{M0} - initial weights of power source and ejected mass, respectively, G_0 -initial weight of body of variable mass), it is required to determines:

1) the optimum law of power decrease N with the corresponding decrease in the power source weight, $G_N = \alpha N$ and
2) the law of the acceleration vector variation due to the jet thrust $\vec{a}$; these laws must provide the minimum time of the displacement between two prescribed points with given velocities.

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x) See Part I, Sections 1-5 (The report at the XII-th Congress of I.A.F., Washington, october, 1961.)

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This problem is described by the set of differential equations

$$\dot{\bar{G}}_M = -\frac{(\bar{G}_M+\bar{G}_N+\bar{G}_\Pi)^2}{\bar{G}_N} \quad a^2 \frac{\alpha}{2g}$$

$$\dot{\vec{r}} = \vec{V} \tag{1}$$

$$\dot{\vec{V}} = \vec{a} + \vec{R}(\vec{r},t)$$

and by boundary conditions:
initial position $(\vec{r}_0, \vec{v}_0)$, final position $(\vec{r}_k \vec{v}_k)$.

In this paper as well as in part 1

$$\bar{G}_n = \frac{G_n}{G_o} = 1 - \bar{G}_{Mo} - \bar{G}_{No}$$ is relative useful

weight; $\bar{G}_N$ and $\bar{G}_M$ - instantaneous values of the relative weight of the power source and that of the reserve of the ejected mass; $\vec{r}, \vec{v}$ - radius-vector and velocity vector of the point; $\vec{R}(\vec{r},t)$ - acceleration vector due to gravitational forces; the values are differentiated with respect to time.

To solve that problem according to Pontrjagin's principle (3), let us write down Hamilton function H

$$H = -P_G \frac{(\bar{G}_M+\bar{G}_N+\bar{G}_n)^2}{\bar{G}_N} a^2 \frac{\alpha}{2g} + \vec{P}_z \cdot \vec{V} + \vec{P}_V \cdot (\vec{a}+R) + P_t \quad (2)$$

Function $H(G_N, \ldots)$ can reach its upper limit with

$$\bar{G}_N = \begin{cases} \bar{G}_M + \bar{G}_n \\ \text{const} \end{cases} \tag{3}$$

For given functional relations $G_N(G_M)$ the variational problem can be divided into two independant variational problems:

a) With the prescribed relative total initial weight of the power source and that of the reserve of the ejected mass we determine the law of decreasing the power source weight; this law consists of extremals /3/ and provides the maximum value of

$$\Phi = -\int_{\bar{G}_{Mo}}^{0} \frac{\bar{G}_N d\bar{G}_M}{(\bar{G}_M+\bar{G}_N+\bar{G}_n)^2} \qquad (4)$$

b) With the prescribed integral functional Φ we determine the minimum time of motion T between two prescribed points with given velocities - this part of the problem is connected with two last equations in the set of equations /1/ and corresponds (as well as with N=const) to the minimum of integral $\Phi = -\frac{\alpha}{2g} \int_0^T a^2\, dt$ if we consider the variational problem with the prescribed time T.

We consider the first problem (a):

It is necessary to determine a piecewise continuous function $G_N(G_M)$, consisting of the pieces $\bar{G}_N=\bar{G}_M+\bar{G}_n$ and $\bar{G}_N$=const which provides maximum $\Phi = \int_0^{\bar{G}_{NO}} \frac{\bar{G}_N d\bar{G}_M}{(\bar{G}_M+\bar{G}_N+\bar{G}_n)^2}$

with a given value of $\bar{G}_n$ and with the following additional condition:

$$\bar{G}_{Mo} + \bar{G}_{No} + \bar{G}_n = 1 \qquad (5)$$

It should be noted, that with $\bar{G}_n \geqslant 0.5$ the extremal $\bar{G}_N(\bar{G}_M)$ cannot involve the direct line $\bar{G}_N=\bar{G}_M+\bar{G}_N$, for in that case condition (5) has no sense. Hence, with $\bar{G}_n \geqslant 0.5$ the optimum law $\bar{G}_N(\bar{G}_M)=\bar{G}_{NO}$=const.

If $\bar{G}_n < 0.5$, the extremal consists of two direct lines:

$$\bar{G}_N=0.25\ ; \quad 0.75-\bar{G}_n \geqslant \bar{G}_M \geqslant 0.25-\bar{G}_n \qquad (6)$$

$$\bar{G}_N=\bar{G}_M+\bar{G}_n; \quad 0.25-\bar{G}_n \geqslant \bar{G}_M \geqslant 0$$

This solution is valid in the range of $0.25 \geqslant \bar{G}_n \geqslant 0$, and in the remainder of the range $\bar{G}_n: 1 \geqslant \bar{G}_n > 0.25$ the extremals are direct lines $\bar{G}_N$=const.

The fi:al results for the functional and the optimum law

G ,M) a:

$$\Phi = \begin{cases} 0.25(1+\ln\frac{1}{4\bar{G}_n}) & \text{with } 0.25 \geqslant \bar{G}_n \geqslant 0 \\ (1-\sqrt{\bar{G}_n})^2 & \text{with } 1 \geqslant \bar{G}_n \geqslant 0.25 \end{cases} \tag{7}$$

$$\bar{G}_N = \begin{cases} 0.25 \text{ with } 0.75 - \bar{G}_n \geqslant \bar{G}_M \geqslant 0.25 - \bar{G}_n;\ \bar{G}_M + \bar{G}_n \text{ with } 0.25 - \bar{G}_n > G_M \geqslant 0 \\ \sqrt{\bar{G}_n} - G_n \end{cases} \tag{8}$$

This optimum relation $1 - \bar{G}_n(\Phi)$ when applying the optimum law of power decrease is compared in Fig.1 with relation $1 - G_n(\Phi)$ for N - const and for multistage decrease of N (n=2), see Fig.1.

7. We consider the dffect of random processes of power decrease on optimum characteristics of the motion of a body of variable mass in a gravitational field.

Under the effect of random factors some elements of the power source can fail at time tj. We state that those elements or the whole power source consists of some independent sections (n - sections) which can be out off when failed;
in a general case this leads to power decrease $N_j = E_j N_0$

$$(0 \leqslant E_j < 1, \quad E_j > E_{j+1}, \quad E_0 = 1).$$

The motion between two instants of time t_j and t_{j+1} takes place with constant power consumption, therefore using equations from section 1 one obtains the following expression for the body weight G_k at the end of motion $t = t_k$ with the constant weight of power source $G_N = \alpha N_0$:

$$G_k/G_0 = \left[1 + \frac{G_0}{2N_0 g} \sum_{j=0}^{\kappa} \frac{1}{E_j} \int_{t_j}^{t_{j+1}} a^2 dt\right]^{-1} \tag{9}$$

where

a - acceleration due to jet thrust

g - Earth's gravitational acceleration .

$\pi < n$ - number of the instants of time when failures take place (it is denoted that $t_{\pi+1} = t_k$)

On the base of the probability characteristics of the processes leading to the failure of power source sections it is possible to determine the averaged values of time t_j x) and to introduce an averaged expression for the functional similar to Φ in section 1.

$$\Phi = \frac{\alpha}{2g} \sum_{j=0}^{\pi} \frac{1}{E_j} \int_{t_j}^{t_{j+1}} a^2 dt \qquad (10)$$

Now, as well as in section 1, we can state the following problem: Let us choose the optimum value of all parameters which influence the specific weight of power source

$\alpha = \alpha(x_j^i n)(i=1,\ldots\ldots,b)$ and the averaged values of $t_j = t_j(x_j^1 \gamma^\ell, \vec{z})$ $\begin{pmatrix} j=1,\ldots C \leq b \\ l=1,\ldots,d \end{pmatrix}$ xx), and also choose such a law $\vec{a}$ (t) and the

initial instant of time t_0, which provide the minimum value of the averaged magnitude Φ xxx) when moving between two prescribed points with given velocities

$(\vec{r}_0, \vec{V}_0)$ and $(\vec{r}_k, \vec{V}_k)$ for the prescribed time T.

Let us consider two kinds of random precesses of failure of power source sections:

a) processes due to heterogeneous external conditions,

b) internal processes or processes due to homogeneous external conditions. Failure probability p_n of one section for an instant of time will be written as

---

x) In this case the probability of simultaneous failures of several sections is neglected.

xx) Restrictions can be imposed on parameters x^1 and γ^e.

xxx) Further we shall consider only averaged values.

$$a)\ p_{nj} = p_{nj}(n,\ x^1,\ \gamma^e)\cdot\rho\,(r,t)$$
$$b)\ p_n = p_{nj}(n, x^1,\ \gamma^e)\cdot\ \ \rho(t)$$

These probabilities determine the instants of time t_j during which power source sections fail.

When moving in a gravitational field /in inertial Cartesian system of coordinates (r^1, r^2, r^3)/it is necessary for minimum Φ to have a trajectory, consisting of functional extremals

$$a)\ J_j = \frac{1}{E_j}\int_{t_j}^{t_{j+1}}\left[\sum_{V=1}^{3}(\ddot{r}^V+R^V)^2 + \lambda_j\,\rho(\vec{r},t)\right]dt, \text{ where } \lambda_j\text{-const}$$
$$b)\ J_j = \frac{1}{E_j}\int_{t_j}^{t_{j+1}}\sum_{V=1}^{3}(\ddot{r}^V+R^V)^2\,dt$$

that is the Euler equations should be valid along the trajectory.

$$a)\ \ddot{a}^i = -\sum_{V=1}^{3} a^V\frac{\partial R^V}{\partial r^i} - \lambda_j\frac{\partial \rho}{\partial r^i} \qquad /j=0,\dots \pi\ ;\ \lambda_\pi = 0/ \tag{11}$$
$$b)\ \ddot{a}^i = -\sum_{V=1}^{3} a^V\frac{\partial R^V}{\partial r^i} \qquad /i=1,2,3/$$

We can see from equation (11) that even with $r_0^i = r_k^i;\ \dot{r}_0^i = \dot{r}_k^i = 0$ and without the gradient of all components of gravitational forces along i-th coordinate $\left(\frac{\partial R^V}{\partial r^i} = 0\right)$ but with function dependence ρ $/\frac{\partial \rho}{\partial r^i} = 0/$ on this coordinate displacement will take place along the coordinate $(r^i \neq r_0^i)$.

Upon equating the sum of first variations of functionals to zero the following conditions for $\vec{a}$ and $\dot{\vec{a}}$ during the instants of time t_j result:

$$\frac{1}{E_j}\ a_j^{V+} = \frac{1}{E_{j-1}}\ a_j^{V-}\ ;\ \frac{1}{E_j}\ \dot{a}_j^{V+} = \frac{1}{E_{j-1}}\ \dot{a}_{j-1}^{V-}\ ; \tag{12}$$

where $\quad a_j^+ = \lim_{\delta\to 0} a(t_j+\delta);\ a_j^- = \lim_{\delta\to 0} a(t_j-\delta);\quad \delta>0$

$$V=1,2,3;\ j=1,\dots,\pi$$

that is, when any section of power source fails the acceleration modulus due to jet thrust and the modulus of acceleration derivative in respect to time must have a discontinuity, these values being decreased by E_j/E_{j-1} times when passing the discontinuity.

From the above condition we obtain the following relations:

$$\text{a)}\begin{cases} \lambda_0 \rho_0 + \sum\limits_{\nu=1}^{3}\left[2\dot{z}_0^{\nu}\dot{a}_0^{\nu+} - (a_0^{\nu+})^2 + 2R_0^{\nu}a_0^{\nu}\right] = \frac{1}{E_\pi}\sum\limits_{\nu=1}^{3}\left[2\dot{z}_k^{\nu}a_k^{\nu-} - (a_k^{\nu-})^2 + 2R_k^{\nu}a_k^{\nu}\right] \\ \rho_j\left(\frac{\lambda_j}{E_j} - \frac{\lambda_{j-1}}{E_{j-1}}\right) + \frac{E_{j-1}-E_j}{E_j^2}\sum\limits_{\nu=1}^{3}(a_j^{\nu+})^2 = 0; \; /j = 1,\dots,\pi-1/ \end{cases} \qquad (13)$$

$$\text{b)}\sum_{\nu=1}^{3}\left\{\frac{1}{E_\pi}\left[2\dot{z}_k^{\nu}\dot{a}_k^{\nu-} - (a_k^{\nu-})^2 + 2R_k^{\nu}a_k^{\nu-}\right] - \left[2\dot{z}_0^{\nu}\dot{a}_0^{\nu+} - (a_0^{\nu+})^2 + 2R_0^{\nu}a_0^{\nu+}\right] - \sum_{j=1}^{\pi}\left(\frac{E_{j-1}}{E_j} - 1\right)(a_j^{\nu+})^2 = 0\right.$$

After solving equations (11) in conjunction with conditions (12) and (13) we choose optimum values of parameters x^1 and γ^e according to the minimum of function Φ.

With large values of n the limit case of continuously decreasing power $\frac{dN}{dt} = -p_n N$ is valid; for this case the solution is greatly simplified.

As an example relations

$$\Phi_{*} = \Phi \Big/ \frac{6 \; (r_k - r_0)^2}{gT^3} \qquad \text{against } p_{*} = p_n T$$

for one-dimensional motion in a forceless field with $V_0 = V_k = 0$ are represented in Fig.2; these relations are obtained on the base of exact solutions for the case of finite number of sections and of continuously decreasing power ($n \rightarrow \infty$).

For comparison curves are plotted in Fig.2 (See dash lines); these curves represent the motion in presence of failures according to the linear law a (t)(dash line in Fig.3) which was optimum in case when failures are absent (see section 4).
An example of relation a (t) which is extremal with account for failures is given in Fig.2 (full line).

8. In sections 1-7 the problems of jet propulsion optimization of a body of variable mass were investigated from the point of view of

nonrelativistic mechanics. It is the ejection velocity of the propellant V that acquires the relativistic level. Let us consider the effect of this phenomenon on the choice of optimum parameters of the motion of a body of variable mass with N=const.

In this case motion and energy equations are

$$P = ma = -\frac{dm}{dt} V \frac{1}{\sqrt{1-v^2/c^2}} \tag{14}$$

$$N = -\frac{dm}{dt} \left(\frac{C^2}{\sqrt{1-v^2/c^2}} - C^2 \right) = maV \frac{1-\sqrt{1-v^2/C^2}}{v^2/C^2} \tag{15}$$

As an example we consider the simplest case of motion with constant thrust P for a given time T(from the point of view of nonrelativistic mechanics this problem was considered in reference (5). In this case the total weight of the required reserve of the ejected mass

$$G_M = -g \int_0^T dm$$

and that of the power source $G_N = \alpha N$

(for the given specific weight α) are:

$$G_M + G_N = PT \left(\frac{g}{V} \sqrt{1-v^2/c^2} + \frac{\alpha}{T} V \frac{1-\sqrt{1-v^2/C^2}}{v^2/c^2} \right) \tag{16}$$

With the prescribed initial weight G_o the maximum of the useful weight $G_n = G_o - (G_M + G_N)$ corresponds to the minimum $(G_M + G_N)$ and to some optimum value V according to equation (16).

Equation (16) can be written in an approximate way as:

$$G_M + G_N \cong PT \left[\frac{g}{V} \left(1 - \frac{v^2}{2c^2}\right) + \frac{\alpha}{2T} V \left(1 - \frac{1}{4} \frac{V^2}{C^2}\right) \right] \tag{17}$$

whence the optimum value is approximately

$$V_{opt} \cong \frac{\sqrt{\frac{2Tg}{\alpha}}}{1 - \frac{5}{4} \frac{Tg}{\alpha c^2}} \cong \frac{\sqrt{\frac{2Tg}{\alpha}}}{1 - \frac{5}{8} V^2/C^2} \tag{18}$$

and G_N/G_M [illegible] $\frac{v^2}{c^2}$ (19)

Hence, the [illegible] effects increase the optimum value of the ejection velocity of the propellant V and the relation G_M/G_N.

We note that with the values T/α of the order 1 year-Kg/kw. the relativistic correction to the unity in the denominator of equation(18) is $\sim 10^{-6}$.

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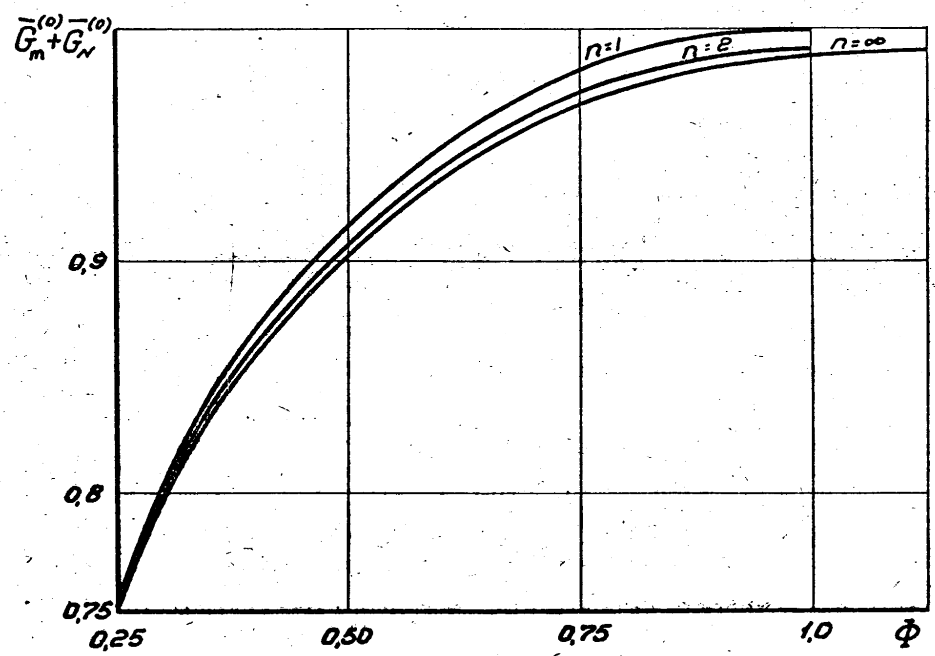

Fig. 1

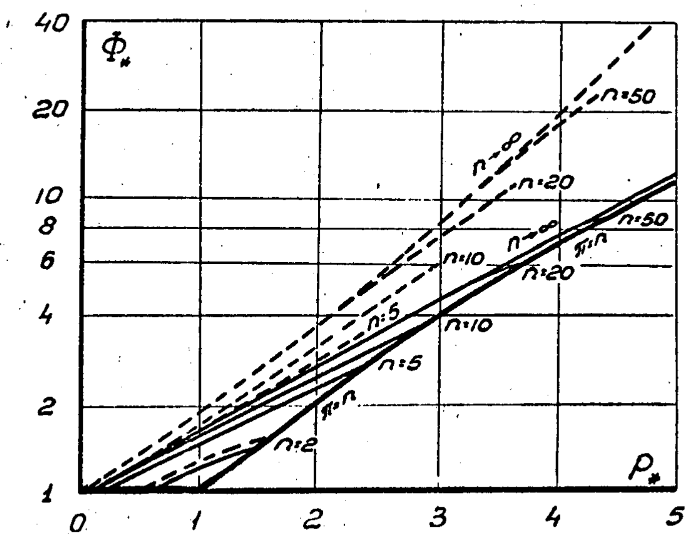
40
Φ_*
20
10
8
6
4
2
1
0
1
2
3
4
5
P_*
n=50
n=∞
n=20
n=10
n=5
n=2
n=50
n=∞
π=n
n=20
n=10
n=5
π=n

Fig. 2

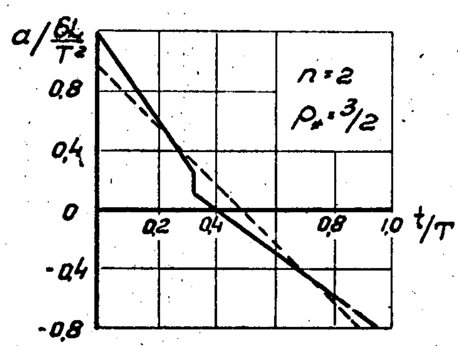
$a/\frac{6L}{T^2}$
0,8
0,4
0
-0,4
-0,8
0,2
0,4
0,8
1,0
t/T
n=2
$P_* = 3/2$

Fig. 3

N. N. MOISEYEV

METHODS OF NON–LINEAR MECHANICS IN THE PROBLEMS OF THE DYNAMICS OF SATELLITES

In the dynamics of satellites and artificial celestial bodies there exist a number of problems which can be described by the following systems:

$$\begin{aligned} x' &= \varepsilon X(x, y, t); \\ y' &= Y_0 (x, y, t) + \varepsilon Y(x, y, t). \end{aligned} \tag{1}$$

Here x is n–dimensional vector, y – m-dimensional vector and ε is the smaller parameter.

In the system (1) some variables change slowly (vector x) and the others rapidly, a circumstance which hampers its numerical integration to say nothing of qualitative analysis of its solutions.

The investigation of the system (1) has many common features with the investigation of systems with a fast–revolving phase which was studied for systems with one degree of freedom by N. N. Bogolyubov and D. N. Zubarev [1]. The basic idea underlying their research was to find such transformation as would separate fast and slow motions. In recent years Soviet scientists have made certain progress in developing effective methods for the investigation of various non–linear problems of the type (1). The investigations V.M.Volosov, G.E.Kuzmak and F.L.Chernousko have been especially fruitful. Some favourable results have been also obtained by the present author. This research provided a deeper insight into the nature of solutions of type (1) systems and brought to the fore a number of ideas which made it possible to simplify in many cases the study of these systems.

In this paper I intend to show several examples of problems pertaining to the dynamics of satellites, which can be reduced to the type (1) system and indicate the ways towards their investigation by asymptotic methods. The main attention will be focussed of the search for such changes of variables which will permit writing the equations in the form convenient for numerical integration.

Inasmuch as the main purpose of this paper is to demonstrate the possibility of asymptotic methods we shall consider only the simplest plane problems although our methods can be also employed for analysis of three dimensional problems without any essential changes.

I. The Problem of the Perturbation of Satellite Orbits

If the orbit of a satellite is considerably removed from the Earth the satellite motion may sometimes be studied within the framework of the perturbation theory since the motion in this case approaches the Keplerian motion. This signifies that the field of the acting forces is closed to the central field and is distorted only by negligible additional magnitudes: the addends which characterise the non—central nature of the Earth's field, small airforces, etc. A satellite in such conditions is in for a long journey amounting to tens and hundreds of turns. The above causes will impart secular perturbations to the parameters of the orbit and it is only natural that we are faced with the task of studying the gradual change in the orbital elements from turn to turn on the assumption that the satellite is a particle. This problem was the subject of numerous investigations on an international scale. We shall indicate here one more approach to this problem employing the methods of non—linear mechanics.

Plane motion of a satellite can be described by the following system of differential equations

$$\left.\begin{aligned} &\frac{d^2u}{d\varphi^2} + u = \frac{\mu}{h^2} - \varepsilon\left\{\frac{f_2\frac{du}{d\varphi}}{u^3h} + \frac{f_1}{u^2h^2}\right\} \\ &\frac{dh}{d\varphi} = \frac{\varepsilon f_2}{u^3h} \end{aligned}\right\} \qquad (1.1)$$

where φ is polar angle; $u = \frac{1}{r}$, r—satellite radius vector; h—areal velocity; $h = r^2\frac{d\varphi}{dt}$, f_1 and f_2—projections of the perturbing force on the radius—vector and transversal. These values are assumed to be arbitrary functions r, φ, $\frac{dr}{d\varphi}$ and h.

At $\varepsilon = 0$ we can integrate the system (1.1) explicitly

$$u = \frac{\mu}{h^2} + c\cos\varphi; \quad h = \text{const.}$$

Let us introduce the new variables (c and ψ) by means of the equations:

$$u = \frac{\mu}{h^2} + c\cos\psi; \quad \frac{du}{d\varphi} = -c\sin\psi.$$

Then the system (1.1) will be reduced to the following system equivalent to it.

$$\left.\begin{aligned}
c' &= \varepsilon\left\{\frac{-f_2 c \sin\psi\, h^5}{(\mu + ch^2\cos\psi)^3} + \frac{f_1 h^2}{(\mu + h^2 c\cos\psi)^2}\right\}\sin\psi + \\
&\quad + \frac{2\mu\varepsilon f_2}{(\mu + h^2 c\cos\psi)^2 h}\cos\psi \equiv F_1(c, h, \psi); \\
h' &= \frac{\varepsilon f_2 h^5}{(\mu + h^2 c\cos\psi)^3} \equiv F_2(c, h, \psi); \\
\psi' &= 1 + \frac{\varepsilon}{c}\left\{\frac{-f_2 c\sin\psi\, h^5}{(\mu + h^2 c\cos\psi)^3} + \frac{f_1 h^2}{(\mu + h^2 c\cos\psi)^2}\right\}\cos\psi + \\
&\quad + \frac{2\mu\varepsilon f_2}{c(\mu + h^2 c\cos\psi)^2 h}\sin\psi \equiv F_3(c, h, \psi).
\end{aligned}\right\} \quad (1.2)$$

The system (1.2) is a system of three equations with respect to the three variables c, h and ψ, the first two changing slowly and the third rapidly. For this reason and basing ourselves on the Bogolyubov—Zubarev idea, we shall look for such transformations which would allow us to separate slow motions from fast motions. That is, we assume that

$$\left.\begin{aligned} c &= a + \varepsilon\xi_1(a, b, x) + \dots \\ h &= b + \varepsilon\eta_1(a, b, x) + \dots \\ \psi &= x + \varepsilon\zeta_1(a, b, x) + \dots \end{aligned}\right\} \quad (1.3)$$

and require that the a, b and x functions satisfy the equations

$$\left.\begin{aligned} a' &= \varepsilon A_1(a, b) + \varepsilon^2 A_2(a, b) + \dots \\ b' &= \varepsilon B_1(a, b) + \varepsilon^2 B_2(a, b) + \dots \\ x' &= 1 + \varepsilon D_1(a, b) + \dots \end{aligned}\right\} \quad (1.4)$$

Substituting the series (1.3) and (1.4) and comparing the coefficients at the same powers of ε and employing only the requirement of the limitedness of the ξ_i, η_i and ζ_i functions we can arrive at any approximations. If we confine ourselves to the first approximation and use the periodicity of the F_i functions with respect to ψ we shall obtain the following system of equations

$$\left.\begin{aligned} & c = a; \quad h = b; \quad \psi = x; \\ & a' = \frac{\varepsilon}{2\pi}\int_0^{2\pi} F_1(a, b, \psi)\,d\psi; \\ & b' = \frac{\varepsilon}{2\pi}\int_0^{2\pi} F_2(a, b, \psi)\,d\psi; \\ & x' = 1 + \frac{\varepsilon}{2\pi}\int_0^{2\pi} F_3(c, h, \psi)\,d\psi. \end{aligned}\right\} \quad (1.5)$$

It will be readily noticed that the first two equations can be integrated independently of the third.

Some special cases should be pointed out.

a) Let the structure of the function f_i be such that the quadratures in the right sides of the system (1.5) are taken explicitly. In this case all is rather simple; the first two equations are very convenient for numerical integration. The third equation is reduced to quadrature. Since the first two equations are redu-

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ced to one first–order equation we can extensively employ the methods of qualitative analysis.

b) Of certain interest are the cases where no quadratures are taken but the problem contains a parameter independent of ε (let, for instance, the initial eccentricity be very small). This makes it possible to use various methods of approximate investigations.

A number of examples pertaining to these cases are examined by Lass and Lorell [2].

c) This approach remains valid also in the general case because when the value of ε is sufficiently small the integration step with respect to t appears very large, at any rate larger than one turn. This means that in constructing a finite–difference scheme we shall have to calculate the quadratures assuming all the variables, except ψ, to be constant. These considerations enable us to develop various methods of numerical integration analogous or identical to the well–know method of double–cycle integration suggested by G.P. Taratynova [3].

The change of variables (1.3) allowed us to separate fast motions from slow motions and perform integration with a larger step with respect to t. However, to attain this we had to calculate additional quadratures in the right sides of the system (1.5). If ε is sufficiently small this complication is compensated for by the amount of the integration step.

If ε is not very small or the accuracy requirements are such that the integration step should be smaller than one turn, the suggested method of solution will no longer be economically feasible and in some cases not applicable at all. For instance, this method can hardly be used to calculate the several last turns of the satellite when the perigee is low and the airodynamic forces are considerable.

II. Problem of Satellite Travel on Last Turns

When the height of the perigee grows small the airodynamic force becomes one of the determining factors. In this case we can neglect the perturbations due to the non–central nature of the

Earth's field. However, another important factor comes to the force. If the satellite is not stabilised by some special method it revolves round the centre of inertia and as it can be of any arbitrary form (for example, the form of dumb–bells) the magnitude of the airodynamic force will essentially depend on the motion of the satellite with respect to the centre of mass. Therefore, in case of low orbits we can no longer separate the motion of the centre of mass from the relative motion.

We shall describe the satellite travel by the following system of differential equations

$$\left.\begin{aligned}
&\frac{dv}{dt} = -\frac{\mu}{r^2}\sin\upsilon + \frac{\rho(r)v^2}{2m}sc_x(v,r,\alpha) \equiv f_1(v,\upsilon,r,\alpha)\\
&\frac{d\upsilon}{dt} = \left(\frac{v}{r} - \frac{\mu}{rv}\right)\cos\upsilon \equiv f_2(v,\upsilon,r);\\
&\frac{dr}{dt} = v\sin\upsilon \equiv f_3(v,\upsilon);\\
&\frac{d\varphi}{dt} = \frac{v}{r}\cos\upsilon \equiv f_4(v,\upsilon);\\
&\frac{d^2\alpha}{dt} + f_5(v,\upsilon,r,\alpha) \equiv 0.
\end{aligned}\right\} \qquad (2.1)$$

Here ρ is atmospheric density, s–characteristic parameter, c_x – dimensionless coefficient, m – mass of the satellite, v – velocity of its centre of mass, υ – angle between the direction of velocity and the local horizon, α – angle of incidence, angle between the velocity vector and a certain direction rigidly connected with the satellite. The function f_s is the aerodynamic moment. If the satellite possesses an axial symmetry, f_s will be the odd function of the variable α.

Let us investigate a case of a revolving satellite. We assume that

$$\frac{d\alpha}{dt} = \omega + z.$$

Let us consider the interesting and important case when ω is large. This means that the period $T = \frac{2\pi}{\omega}$ is small as compared to the characteristic time scale of the change of the other parameters.

We assume $\frac{1}{\omega} = \varepsilon$ and introduce the new independent variable

$$\tau = \omega t.$$

Then the system of equations (2.1) will be reduced to the following form:

$$\left.\begin{array}{l} \frac{dv}{d\tau} = \varepsilon f_1; \quad \frac{d\upsilon}{d\tau} = \varepsilon f_2; \quad \frac{dr}{d\tau} = \varepsilon f_3; \\ \frac{d\varphi}{d\tau} = \varepsilon f_4; \frac{dz}{d\tau} = -\varepsilon f_5; \quad \frac{d\alpha}{d\tau} = 1 + \varepsilon z. \end{array}\right\} \quad (2.2)$$

The system of equations (2.2) is analogous to the system (1.2). It will be natural therefore to change the variables used in the preceding section:

$$\left.\begin{array}{l} v = u + \varepsilon v_1(u, \theta, R, \zeta, \beta) + \dots \\ \upsilon = \theta + \varepsilon \upsilon_1(u, \theta, R, \zeta, \beta) + \dots \\ r = R + \varepsilon r_1(u, \theta, R, \zeta, \beta) + \dots \\ z = \zeta + \varepsilon z_1(u, \theta, R, \zeta, \beta) + \dots \\ \alpha = \beta + \varepsilon \alpha_1(u, \theta, R, \zeta, \beta) + \dots \end{array}\right\} \quad (2.3)$$

where u, θ, R, ζ and β satisfy the equations

$$\left.\begin{array}{l} u' = \varepsilon A_{11}(v, \theta, R, \zeta) + \varepsilon^2 A_{12} + \dots \\ \theta' = \varepsilon A_{21} + \varepsilon^2 A_{22} + \dots \\ R' = \varepsilon A_{31} + \varepsilon^2 A_{32} + \dots \\ \zeta' = \varepsilon A_{41} + \varepsilon^2 A_{42} + \dots \\ \beta' = 1 + \varepsilon B_1 + \dots \end{array}\right\} \quad (2.4)$$

If we confine ouselves to the first approximation we shall come to the following system:

$$\left.\begin{aligned} &\frac{dv}{d\tau} = \frac{\varepsilon}{2\pi}\int_0^{2\pi} f_1\,d\alpha;\\ &\frac{d\upsilon}{d\tau} = \varepsilon f_2;\quad \frac{dr}{d\tau} = \varepsilon f_3;\\ &\frac{dz}{d\tau} = -\frac{\varepsilon}{2\pi}\int_0^{2\pi} f_5\,d\alpha;\\ &\frac{d\alpha}{d\tau} = 1 + \varepsilon z;\quad \frac{d\varphi}{d\tau} = \varepsilon f_4. \end{aligned}\right\} \qquad (2.5)$$

The integral in the first equation of this system is not equal to zero

$$\frac{1}{2\pi}\int_0^{2\pi} f_1\,d\alpha = -\frac{\mu}{r}\sin\upsilon + \frac{\rho v^2}{2m}s\bar{c}_x(v, r) \equiv \bar{f}_1$$

$$\bar{c}_x = \frac{1}{2\pi}\int_0^{2\pi} c_x(v, r, \alpha)\,d\alpha.$$

Therefore, to a first approximation it is sufficient to replace in the system of equations (2.1) the coefficient c_x by its mean value with respect to the angle of incidence. It follows from this that our problem is reduced to the integration of the following equations:

$$\frac{dv}{dt} = \bar{f}_1 \qquad \frac{d\upsilon}{dt} = f_2 \qquad \frac{dr}{dt} = f_3.$$

The other equations are integrated in this case in quadratures.

Due to the decreasing of the velocity the height of the osculating perigee will all the time be decreasing and the satellite will pass through more denser atmospheric layers. The lift force will become considerable as well as the air friction. The last circumstance will decrease the angular velocity of rotation and the parameter ε will cease to be small. The further descent of the spaceship will entail cessation of the rotary motion and the transition to the phase of the oscillatory motion. The quite different theory is needed for its description.

III. Problem of Satellite Travel at the End of the Last Turn

Assume that satellite is a symmetrical body with a large reserve of static stability. This assumption is very important. It will underlie the construction of the methods of asymptotic integration. Let us assume that the satellite travel over the last portion is described by the following system of differential equations:

1. The equations of momentum

$$\left.\begin{aligned}\frac{dv}{dt} &= -g\sin\theta - \frac{\rho v^2}{2m} s c_x \equiv f_1(v,\theta,y,\alpha);\\ \frac{d\theta}{dt} &= -g\frac{\cos\theta}{v} + \frac{\rho v}{2m} s c_y \equiv f_2(v,\theta,y,\alpha).\end{aligned}\right\} \quad (3.1)$$

2. Kinematic relations

$$\left.\begin{aligned}\frac{dx}{dt} &= v\cos\theta \equiv f_3(v,\theta)\\ \frac{dy}{dt} &= v\sin\theta \equiv f_4(v,\theta).\end{aligned}\right\} \quad (3.2)$$

3. The equations of moments

$$\frac{d^2\alpha}{dt^2} + \frac{d^2\theta}{dt^2} + \frac{\rho v^2}{2I} s m_z(v,y,\alpha) + \frac{\rho v}{2I} s l m_z^{\omega}\left(\frac{d\alpha}{dt} + \frac{d\theta}{dt}\right) = 0. \quad (3.3)$$

For simplicity we assume in this problem that the gravitational field is homogeneous (rejection of this assumption will not change our way of consideration. g is the gravity acceleration; θ—angle between the velocity vector and the x axis pointed along the horizon, the y axis will be directed vertically up; $c_y = c_y(v, y, \alpha)$ – dimensionless coefficient of the lift force; $m_z = m_z(v, y, \alpha)$ – dimensionless coefficient of the aerodynamic moment $m_z^{\omega} = m_z^{\omega}(v, y)$ – multiplier at the value of the angular velocity in the dimensionless coefficient of the damping moment (the damping moment is naturally regarded as a linear function of the angular velocity); I — moment of inertia, l — length factor. The other designations are the same as in the preceding sections.

Let us use the equation (3.1) to eliminate the derivatives $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$ from the equation (3.3). We shall obtain the following equation

$$\frac{d^2\alpha}{dt^2} + f_5^*(v, \theta, y, \alpha) + f_6 \frac{d\alpha}{dt} = 0.$$

High frequence oscillation is one of the important features of the motion of a statically stable body on the last portion of the path. It means that during one oscillation the other parameters of the system will change negligibly.

We assume that

$$f_5^* = \lambda^2 f_5$$

where λ is a large parameter. We now replace the independent variable

$$t = \frac{\tau}{\lambda} = \varepsilon\tau.$$

Then the system (3.1 – 3.3) will take the following form:

$$\left.\begin{aligned} &\frac{dv}{d\tau} = \varepsilon f_1; \quad \frac{d\theta}{d\tau} = \varepsilon f_2; \quad \frac{dx}{d\tau} = \varepsilon f_3; \quad \frac{dy}{d\tau} = \varepsilon f_4; \\ &\frac{d^2\alpha}{d\tau^2} + f_5(v, \theta, y, \alpha) + \varepsilon f_6(v, \theta, y)\frac{d\alpha}{d\tau} \quad 0. \end{aligned}\right\} \quad (3.4)$$

When $\varepsilon = 0$ the system (3.4) will appear as:

$$\alpha'' + f_5(v, \theta, y, \alpha) = 0 \quad v = \text{const}, \ \theta = \text{const}, \ y = \text{const}. \qquad (3.5)$$

We shall assume that the solution of the system (3.5) is known

$$\left.\begin{aligned} \alpha &= q(c, \psi, v, \theta, y) \\ \psi &= \omega(c)(\tau + \tau_0). \end{aligned}\right\} \qquad (3.6)$$

Since the equation (3.5) describes the oscillation the function q will be the periodic function of the variable ψ. Denote the period by T (without breaking community $T = 2\pi$). Various methods can be used for the practical calculation of the function (3.6). Thus, for example, if the angle of incidence is small the function q can be found by the methods of the quasilinear theory. If α is large various approximations can be employed which in combination with the methods of the small parameter or averaging operation will allow us to calculate the function q to a sufficient accuracy. Let us determine the derivative α' through the equation

$$\alpha' = \omega q_{\psi}. \tag{3.7}$$

The equations (3.6) and (3.7) determine the new derivatives c and ψ which are used to reduce the system (3.4) to the following form:

$$\left.\begin{aligned}
&\frac{dv}{d\tau} = \varepsilon f_1;\ \frac{d\theta}{d\tau} = \varepsilon f_2;\ \frac{dy}{d\tau} = \varepsilon f_4;\ \frac{dx}{d\tau} = \varepsilon f_3;\\
&\frac{dc}{d\tau} = -\frac{\varepsilon}{\Delta}\{f_6 \omega q_{\psi}^2 + \Phi_1 \omega q_{\psi\psi} - \Phi_2 q_{\psi}\};\\
&\frac{d\psi}{d\tau} = \omega(c, v, \theta, y) + \frac{\varepsilon}{\Delta}\{f_6 q_{\psi} q_c + \Phi_1(\omega_c + q_{\psi c}) -\\
&\quad - \Phi_2 \omega q_{\psi\psi}\} = \omega + \varepsilon\Psi.
\end{aligned}\right\} \tag{3.8}$$

where

$$\Delta = [q_c \omega q_{\psi\psi} - (\omega_c + q_{\psi c}) q_{\psi}];$$

$$\Phi_1 = q_v f_1 + q_{\theta} f_2 + q_y f_4;$$

$$\Phi_2 = q_{\psi v} f_1 + q_{\psi\theta} f_2 + q_{\psi y} f_4.$$

It is easy to show that the value of Δ does not depend on the variable ψ. The system (3.8) is a system of six first–order equations with respect to six unknown variables of which the first five change slowly. Therefore, we once more obtain the problem which was considered above.

We no longer need to write the standard change of the variables. We only confine ourselves to writing the system of equations which should be numerically integrated if we deal exclusively with the equations of a first approximation

$$\frac{dv}{d\tau} = \varepsilon\{-g\sin\theta - \frac{\rho v^2 s}{4\pi m}\int_0^{2\pi} c_x[v, y, q(c, \psi, v, y, \theta)]d\psi\} \qquad (3.9_1)$$

$$\frac{d\theta}{d\tau} = \varepsilon\{-\frac{g\cos\theta}{v} + \frac{\rho v s}{4\pi m}\int_0^{2\pi} c_y[v, y, q(c, \psi, v, y, \theta)]d\psi\} \qquad (3.9_2)$$

$$\frac{dx}{d\tau} = \varepsilon v\cos\theta \qquad (3.9_3)$$

$$\frac{dy}{d\tau} = \varepsilon v\sin\theta \qquad (3.9_4)$$

$$\frac{dc}{d\tau} = -\frac{\varepsilon f_6 \omega}{2\pi\Delta}\int_0^{2\pi} q_\psi^2 d\tau - \frac{\varepsilon}{2\pi\Delta}\int_0^{2\pi}[\Phi_1 \omega q_{\psi\psi} - \Phi_2 q_\psi]d\psi \qquad (3.9_5)$$

$$\frac{d\psi}{d\tau} = \omega + 0(\varepsilon). \qquad (3.9_6)$$

Consequently, we again face the problem of integration of a system of four differential equations (3.9_1), (3.9_2), (3.9_4) and (3.9_5) after which the equations (3.9_3) and (3.9_6) are integrated in quadratures.

IV. Resonance Problems in the Theory of Satellites

Interesting resonance problems arise from the study of the satellite travel relative to the centre of mass. If the orbit is sufficiently high it can be assumed that motion with respect to the centre of mass does not effect the motion of the centre of mass itself. Moreover, to simplify our calculations, we shall exclude from consideration the perturbing factors and assume that the centre of gravity moves along an ellipse whose eccentricity is e. V.V.Beletsky [4] has shown that the plane motion of the satellite is described in this case by the following equation:

$$(1+e\cos\theta)\frac{d^2\delta}{d\theta^2}-2e\sin\theta\frac{d\delta}{d\theta}+3a^2\sin\delta=4e\sin\theta \qquad (4.1)$$

Here $a=\frac{A-C}{B}$; B – moment of inertia relative to that axis of the satellite which always remains perpendicular to the orbit plane; A and C – moments of inertia relative to the axes in the plans of the orbit and $A > C$ and $a \leq 1$; $\delta = 2\upsilon$, υ – angle between the radius vector of the centre of inertia and the axis of inertia with respect to which the moment equals C; θ – angular distance of the radius vector from the orbit perigee.

If the orbit is circular ($e = 0$) the equation (4.1) is integrated in elliptic functions. Of the two positions of equilibrium the position $\delta = \upsilon = 0$ is stable.

If $e \neq 0$ the asymptotic methods can be used to investigate two cases: 1) $e \ll 1$ – the orbit is nearly circular; 2) $a \ll 1$ – the form of the satellite is nearly axisymmetric. These cases have been examined by F.L.Chernousko.

1) Let $e \ll 1$, then the equation (4.1) can be rewritten in the following form:

$$\delta'' + 3a^2\sin\delta = e(4\sin\theta + 3a^2\cos\theta\sin\delta + 2\sin\theta\delta') + 0(e^2). \quad (4.2)$$

The solution of the equation (4.2) when $\varepsilon = 0$ is denoted

$$\delta = q(c,\varphi) \qquad \delta' = \omega(c)q_\varphi$$

where $\varphi = \theta + \theta_0$.

We shall employ the way of consideration used in the preceding section to reduce the equation (4.2) to the following system of differential first–order equations:

$$c' = e\Phi_1(\theta, \theta_0 c) \qquad \theta_0' = \omega(c) - 1 + e\Phi_2(\theta, \theta_0 c). \quad (4.3)$$

c^* will be the root of the equation

$$\omega(c^*) = 1.$$

If $c - c^* = 0(e)$ the right sides of the system (4.3) can be investigated by the methods used in the preceding sections (for this purpose it will be sufficient, for example, to add the "fast" unknown value $\frac{d\theta}{dt} = 1$); we shall then come to the following system of equations containing no θ:

$$c' = e\bar{\Phi}_1(\theta_0 c); \quad \theta_0' = \omega(c) - 1 + e\bar{\Phi}_2(\theta_0 c) \qquad (4.4)$$

From the equations $c' = \theta_0' = 0$ we find stationary resonance solutions. If we investigate the stability by the usual methods we shall obtain the classical result: that position of the satellite equilibrium will be stable at which in perigee the axis of inertia C is pointed along the radius vector.

2) When $a \ll 1$, F. L. Chernousko specially selected the variables to reduce the problem to the investigation of the equation which after the averaging operation took the form

$$\frac{d^2 2\alpha}{d\tau^2} + 3a^2 F(e) \sin 2\alpha = 0$$

where $F(e)$ is a monotonically decreasing function and $F(e) > 0$ if $e > e_0 \sim 0.68$ and $F(e) < 0$ at $e < e_0$.

It follows from this that when eccentricities are large there takes place a replacement of stable and unstable equilibrium positions and that position becomes stable at which the axis of minimal moment of inertia of the satellite is perpendicular to the radius vector in the perigee.

Conclusion

The following should be pointed out in conclusion:

1) In our examination of the general scheme of the "separation of motions" we confined ourselves in all problem to constructing a first approximation. We can indicate a number of problems where this approach will be clearly inadequate. However, the procedure oulined in this paper can also be used for a more detailed analysis.

2) The above problems have been specially selected to show the versatile character of the problems of satellite dynamics for which the methods of asymptotic integration can serve the basis for research.

From the studies of the works of de Sparre'a, Captain of French Artillery (late 19th century), down to the well-known research of D.A.Ventcel and V.S.Pugachev (thirties of the 20th century), specialists in exterior ballistics concentrated on developing the methods of asymptotic integration. I believe that it is precisely these methods that served the rational basis of success attained in the solution of the general problem of exterior ballistics. I am convinced that this can in full measure be applied also to the dynamics of satellites.

There is a broad class of problems whose destiny will depend on the progress in the application of asymptotic methods. The problems that arise in the dynamics of satellites are undoubtedly far more complicated than the problems of ballistics but the methods of research become now more effective.

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INVESTIGATION OF THE DYNAMICS OF GRAVITATIONAL STABILIZATION SYSTEM

(Abstract)

The paper is devoted to theoretical investigation of the dynamics of the satellite gravitational stabilization system. The principle scheme of the system under investigation is described in (1). In this scheme a second body called a stabilizer is connected to the satellite body by means of a spherical gimbal. The position of the stabilizer relative to the satellite is fixed by springs. The relative mobility of the satellite and stabilizer makes it possible to introduce linear damping in the system, for instance, by means of a magnetic damper.

Equations of motion of the satellite-stabilizer system are derived in assumption that motion takes place in the central Newtonian gravitational field in a medium without resistence. The satellite orbit is an ellipse fixed in absolute space. Thus, the study does not take into account the mutual dependence of the system's onward and rotational motions, and the influence of perturbing factors is ignored.

The work consists of six paragraphs. In §1 expressions are obtained for kinetic energy and force function of the satellite-stabilizer system without any limitations on inertia characteristics of the satellite and stabilizer. In §2 equations of motion of the system are derived linearized in the vicinity of the stable equilibrium position. These equations are reduced to a dimensionless form for the concrete stabilizer design.

In §3 Lyapunov's function is constructed by means of the generalized energy integral, and necessary and sufficient conditions of the asymptotic stability of the equilibrium position of the satellite-stabilizer system in a circular orbit are obtained. In §4 the influence of the orbit ellipticity on the system behaviour is analyzed. Formulas for amplitudes of the satellite and stabilizer forced oscillations are derived with an accuracy to the square of eccentricity.

The problem of the rapidity of attenuation of natural oscillations of the satellite-stabilizer system is considered in §5. As a criterion of the minimum duration of the transition process a condition is selected of the maximum of the real part (which is the least in modulus) of the roots of the system characteristic equation. In §6 an example is given of the attenuation of the system natural oscillations in circular and elliptical orbits.

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GRAVITATIONAL STABILIZATION SYSTEM OF ARTIFICIAL SATELLITES

Abstract

A purely passive scheme of a gravitational stabilization system of artificial satellites is suggested which uses the properties of the Earth's gravitational field. The main dynamic characteristics of this system are briefly indicated.

The realization of scientific investigations in interplanetary space by means of artificial satellites often requires precise three-axis or one-axis orientation of a satellite with respect to the Earth for a long period of time. The use of active systems of orientation at considerable lifetime of the satellite leads to a number of difficulties connected with the large energy or propellant consumption, weight and complexity of these systems.

It is possible to create passive systems of stabilization on the basis of the use of magnetic and gravitational fields, effects of light pressure, the drag of the atmosphere, etc. The important positive property of passive systems consists in the fact that these systems can function for a long period of time without energy or propellant consumption. The most essential drawback of passive systems is the relatively small magnitude of controlling moments.

In the present paper a possibility is considered of the satellite stabilization with respect to a thrihedron formed by the radius-vector, transversal and binormal to the orbit. This thrihedron will be called the orbital coordinate system. The principle of stabilization is based on the use of the property of the Newtonian gravitational field to definitely orient a body moving in it. This body has unequal moments of inertia relative to its main central axes.

If a satellite moves in the central Newtonian field of forces along the circular orbit, there are four stable positions of relative equilibrium corresponding to the coincidence of the major axis of the satellite inertia ellipsoid with radius-vector and of the minor axis with the binormal to the orbit (1,2) (Fig.1). The stable equilibrium positions turn to each other at the satellite turn by 180° around the radius-vector and the binormal to the orbit. In an absolute coordinate system the satellite rotation around the binormal to the orbit with an angular velocity equal to the angular velocity of the satellite mass centre motion in orbit corresponds to the relative equilibrium position.

In the absence of inner energy dissipation the value of the amplitudes of the satellite small oscillations about the equilibrium position does not vary with time. The accuracy of stabilization is determined by the initial values of angles and of angular velocities of the satellite. The introduction of dissipative forces to the system transforms the satellite stable relative equilibrium positions to the asymptotically stable ones. Then the amplitudes of natural oscillations caused by the initial values of angles and angular velocities tend to zero.

The simplest scheme which makes it possible to introduce dissipative forces and to stabilize oscillations of an artificial satellite with respect to the orbital coordinate system in a circular orbit is illustrated in Fig.1. A central spherical cavity filled with viscous fluid is inside the gravitationally stable satellite. The satellite occilatory motion leads to replacement of viscous fluid relative to the satellite body and to energy dissipation. The sphere in Fig.1 can be replaced by a cavity formed by two spherical envelopes. There is optimum viscosity ensuring the maximum velocity of dissipation of the oscillations energy for a given thickness of the layer and the density of viscous fluid enclosed between spherical envelopes.

The main drawback of the scheme of damping of the satellite oscillations by means of viscous fluid is that large quantity of fluid is required for relatively rapid energy dissipation, since it turns out that in the optimum case of damping the moment of inertia of fluid should be comparable in magnitude with the satellite maximum moment of inertia. The effectiveness of this scheme is somewhat increased of fluid is placed in an enclosed toroidal volume situated outside the satellite.

In 1956 D.E.Okhotsimsky suggested a more effective scheme of stabilization and damping. This scheme is presented in Fig.2. By means of a spherical gimbal P a second body called a stabilizer is connected to the satellite body. The stabilizer represents two bars equal in length rigidly connected with each other with equal weights at the ends. The coordinate systems $O_1x_1y_1z_1$ and $O_2x_2y_2z_2$ are the main central thrihedrons connected with the satellite and the stabilizer, respectively. The position of the stabilizer relative to the satellite body is fixed by centering springs.

The stabilizer parameters (the length of bars, weight, the angle between bars) are chosen with a view to ensuring the gravitational stability of the satellite-stabilizer system at the rigid fixation of the stabilizer relative to the satellite. In the stable equilibrium position of the satellite-stabilizer system the bars are located in the orbital plane, $O_1x_1 \parallel O_2x_2$, $O_1y_1 \parallel O_2y_2$ and is parallel to the tangent to the circular orbit, $O_1z_1 \parallel O_2z_2$.

Nonrigid fixation of the mutual position of the satellite and stabilizer by means of elastic connection is realized with a view to introducing linear damping members to the system using the relative mobility of the satellite and stabilizer. Practical realization of linear damping in the satellite-stabilizer system is possible, for instance, by means of a magentic damper whose action is based on the use of Foucault currents or a liquid damper. Such dampers are widely used in construction of instruments.

The proposed scheme makes it possible to ensure the satellite stabilization relative to the orbital coordinate system at any inertia characteristics of the satellite. The shape of the satellite is of no importance in a medium without resistance. Motion of the system is determined by inertia characteristics of the satellite and stabilizer and by the coordinates of a spherical gimbal with respect to the thrihedrons $O_1x_1y_1z_1$ and $O_2x_2y_2z_2$.

Moments of inertia of the stabilizer are proportional to the square of the length of the bars and the maximum size of the bars is determined only by the requirements of the design rigidity. Therefore the ratio between the moments of inertia of the satellite and stabilizer necessary for the satisfactory transition process is easily ensured by means of small masses at the end of bars at the expense of increasing their lengths. From the design point of view folding or telecopic bars (or bars from metallic strips) which are rolled up under the action of resilience forces appear to be convenient.

We can abstain from introducing elastic connection between the satellite and stabilizer if the satellite is gravitationally stable without the stabilizer, and the gimbal is located on the major central axis corresponding to the satellite moment of inertia average in magnitude. In this case the role of fixing elastic connections is played by gravitational moments.

The scheme of the satellite-stabilizer system in Fig.2 is the most simple and at the same time the most general since it solves the stabilization problem at any parameters of the satellite. Consideration of a more complex forms of the stabilizer does not add anything new to this scheme.

Up till now we spoke about motion of the satellite-stabilizer system in a circular orbit in a medium without resistance. On an elliptic orbit forced eccentricity oscilations caused by nonuniformity of rotation of the orbital coordinate system are added to natural oscillations decreasing in amplitude. Eccentricity oscillations take place in the orbital plane.

The amplitude of eccentricity oscillations is proportional to the value of the orbit eccentricity and depends on inertia characteristics of the satellite and stabilizer. The frequency of eccentricity oscillations coincides with the frequency of the revolution of the centre of mass of the satellite-stabilizer system in orbit and, consequently, the angle of rhe satellite deviation relative to the orbital coordinate system changes very slowly in time. Eccentricity oscillations can be easily calculated and can be taken into account at processing the results of experiments carried out at the satellite.

At motion of the satellites in orbits with heights lower than 600 km it is necessary to take into account the influence of the atmosphere which in the main is reduced to forces of resistance applied at the centres of pressure of the satellite and stabilizer and aimed against the velocity of the centre of mass of the satellite-stabilizer system.

The gravitationally stable scheme of the satellite-stabilizer system will be at the same time aerodynamically stable at the unchangeable equilibrium position of the satellite and stabilizer relative to the orbital coordinate system if the following conditions are satisfied:

1) Axes O_1 P and O_2 P (Fig.2) are axes of geometrical symmetry of the satellite and stabilizer.

2) Neither satellite, nor stabilizer should be aerodynamically unstable.

3) The stabilizer aerodynamic braking (the resistance force to mass ratio) is not higher than the satellite aerodynamic braking, i.e. the satellite fulfills the role of a parachute with respect to the stabilizer.

The formulated conditions are sufficient. Taking into account gravitational stability of the satellite-stabilizer system and the existence of elastic connection these conditions can be weakened.

In a circular orbit the atmospheric resistance leads to the increase of frequencies of natural oscillations of the satellite-stabilizer system. The second effect of the action of the atmoshpheric drag on the system oscillations is connected with the carrying along of the atmosphere by the rotating Earth and depends on the orbital inclination and height and the position of the centres of pressure of the satellite and stabilizer.

The maximum amplitude of forced oscillations due to the rotation of the atmosphere does not exceed several degrees and decreases with height, as calculations have shown. The frequency of oscillations coincides with the frequency of the rovolution of the system centre of mass in the orbit. The atmosphere rotation does not influence the oscillations in the orbital plane and effects only the system oscillations which drives it out of the orbital plane.

In an elliptic orbit the influence of the Earth's atmosphere on motion of the satellite-stabilizer system is more complicated which is connected with the variation of the atmosphere density with height.

With sufficiently good knowledge of aerodynamic forces acting on the satellite and stabilizer the oscillations caused by the action of the atmosphere can be calculated and taken into account.

It should be pointed out that in principle it is possible to exclude the influence of the atmospheric resistance on the oscillations of the satellite-stabilizer system. For this purpose it is sufficient to make the satellite and stabilizer aerodynamically neutral and to ensure the equality of values of their aerodynamic braking. When these conditions are satisfied the atmospheric resistance affects only the onward motion of the satellite-stabilizer system and does not influence the system oscillatory motion.

The demand to stabilize the satellite in a prescribed stable equilibrium position imposes limitations on the initial conditions of the satellite after its separation from the last stage of the rocket carrier. The values of angles and angular velocities of the satellite should be such that in the process of quieting the transition from one stable equilibrium position to the other will be excluded. If this condition is not fulfilled, then the gravitational stabilization system should be introduced into the operating range by means of the active system of quieting which decreases the initial amplitudes to the necessary value. The decrease of the initial angular velocity of the system is also possible at the expense of increasing its moments of interita in the process of opening of the stabilizer bars which were located before the placing of the satellite to the orbit in the folding state.

The proposed gravitational stabilization system can function for a long period of time and does not require a the energy consumption for stabilization. The accuracy of the satellite stabilization is determined only by the accuracy of the manufacture of the satellite-stabilizer system and in principle can be high without any limitations. The weight of the stabilizer ensuring the optimum transition process does not exceed several per cent of the satellite weight if the length of the bar is equal to the satellite double maximum linear size.

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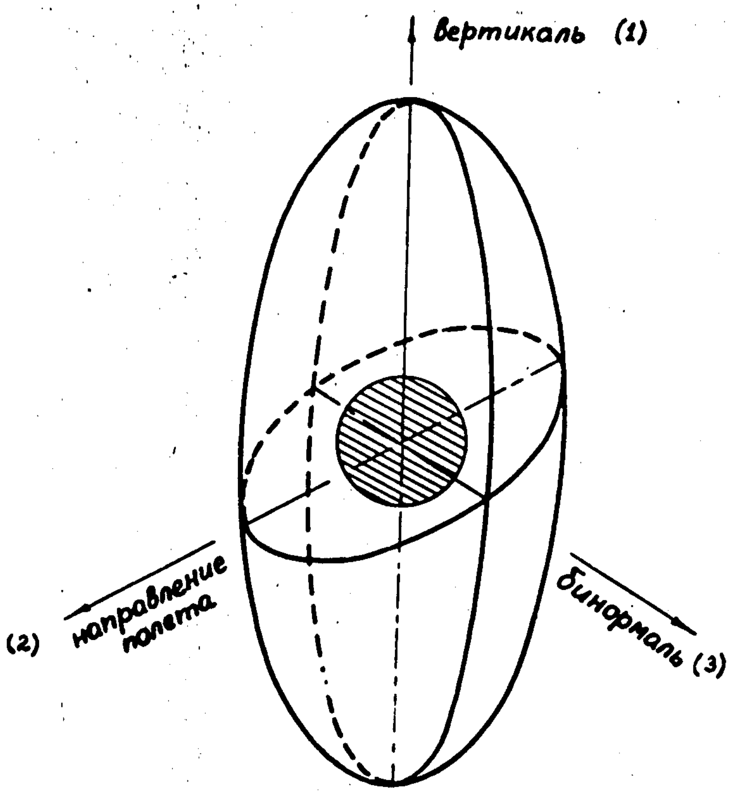

Fig. 1

1. the vertical
2. the direction of the flight
3. the binormal

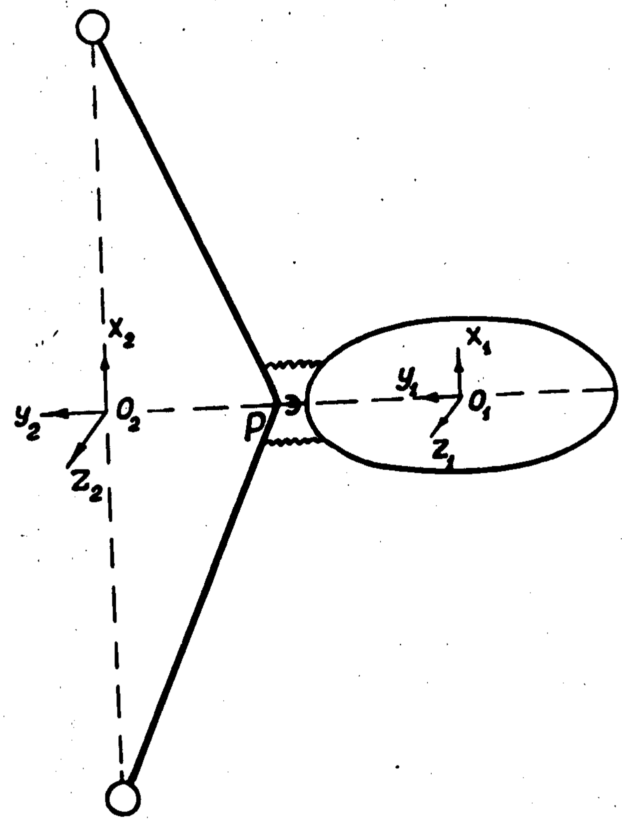

Fig. 2. The scheme of the satellite-
- stabilizer system.

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